Seshadri constants at very general points

Michael Nakamaye

February 1, 2008

0 Introduction

The goal of this paper is to study the Seshadri constant of an ample line bundle A at a very general point η of a smooth projective variety X.

Definition 0.1 *Suppose X is a smooth projective variety, $x \in X$, and A an ample line bundle on X. Then we define the Seshadri constant of A at x by*

$$\epsilon(x, A) = \inf_{C \ni x} \frac{c_1(A) \cap C}{\mathrm{mult}_x(C)};$$

here the infimum runs over all integral curves $C \subset X$ passing through x.

Equivalently, if $\pi : Y \to X$ denotes the blow–up of X at x with execptional divisor E, then

$$\epsilon(x, A) = \sup_r \{r \in \mathbf{Q}^+ \mid \pi^*(A)(-rE) \text{ is ample}\}.$$

The Seshadri constant $\epsilon(x, A)$ measures how many jets nA separates at η asymptotically as $n \to \infty$. In the case when X is a surface, it is known [EL] that $\epsilon(\eta, A) \geq 1$. Meanwhile, Ein, Küchle, and Lazarsfeld [EKL] have established a lower bound in arbitrary dimension

$$\epsilon(\eta, A) \geq \frac{1}{\dim X}.$$

The factor of $\dim X$ appearing in the general result is a function of the "gap argument" used in the proof. The same gap argument is also responsible for the presumably extra factor of $\dim X$ in the known results for global generation of adjoint bundles (see [S] for example).

Our goal in this paper is to make some progress toward obtaining better lower bounds for $\epsilon(\eta, A)$. Ultimately, however, one would like a bound which does not depend on the dimension of X and so from our point of view the interest of this work lies more in the methods used than in the explicit results. The basic idea employed here, namely that a singular Seshadri exceptional subvariety influences the dimension count for sections with specified jets, was already presented in [N]. The counting is difficult and we have tried to find a compromise between computational complexity versus obtaining the best possible results. Thus we have counted very carefully in the three–fold case and less so in the higher dimensional case.

In order to clarify our basic strategy we will go through the argument completely in the three–fold case to obtain:

Theorem 0.2 *Suppose X is a smooth three–fold and $\eta \in X$ a very general point. Then for any ample line bundle A on X we have*

$$\epsilon(\eta, A) \geq \frac{1}{2}.$$

The proof of Theorem 0.2 will use the work of Ein and Lazarsfeld [EL] on surfaces as well as the uniform bounds on symbolic powers obtained in [ELS]. We will then establish the following slight quantitative improvement of the main result in [EKL] for arbitrary dimension:

Theorem 0.3 *Suppose X is a projective variety of dimension $d \geq 4$ and A an ample line bundle on X. Then for a very general point $\eta \in X$ we have the lower bound*

$$\epsilon(\eta, A) > \frac{3d+1}{3d^2}.$$

The proofs of Theorem 0.2 and Theorem 0.3 follow [EKL] line for line, our only innovation coming in counting jets. The fundamental observation is that if $\epsilon(\eta, A)$ is small this puts restrictions on $h^0\left(X, nA \otimes m_\eta^{n\alpha}/m_\eta^{n\alpha+1}\right)$ for various values of α. The smaller $\epsilon(\eta, A)$ becomes the greater these restrictions are. In [EKL] the lower bound on the multiplicity which can be imposed at η comes from the Riemann–Roch theorem. We improve upon this here by considering the above mentioned obstructions and this translates into a better lower bound for $\epsilon(\eta, A)$. The main difficulty in establishing the lower bound $\epsilon(\eta, A) \geq \frac{1}{d-1}$ in general is that the set of points where the bound $\epsilon(\eta, A) \geq \frac{1}{d}$ from [EKL] fails may contain divisors once $d \geq 3$. The reader only interested in the method employed and not the details of counting should skip to §2 after going through Lemma 1.3.

Finally, we would like to point out the similarity between the counting methods used here and those employed by Faltings and Wüstholz [FW] to reprove and extend the Schmidt subspace theorem. In particular, the measure theoretic aspect of [FW] is very closely related to our counting of jets. The asymptotic dimensions $h^0\left(X, nA \otimes m_\eta^{n\alpha}/m_\eta^{n\alpha+1}\right)$, as $n \to \infty$, can be used to define a measure μ on $[0, m(A)]$, with $m(A)$ defined in §2. Here $\mu((a,b))$ would measure the asymptotic cost of raising the multiplicity at η from a to b.

Notation and Conventions

- All varieties considered will be defined over the complex numbers $\mathbf{C}$.
- A point x of an irreducible variety X will be called very general if x belongs to the complement of countably many closed, proper subvarieties.
- If $x \in X$ then $m_X \subset \mathcal{O}_X$ is the maximal ideal sheaf of x.

- Suppose $V \subset X$ is an irreducible subvariety and $s \in H^0(X,L)$ then $\mathrm{ord}_V(s)$ is the order of vanishing of s along V.
- If L is a line bundle $\mathrm{BS}(L)$ denotes the stable base locus of L, that is

$$\mathrm{BS}(L) = \{x \in X : s(x) = 0 \text{ for all } s \in H^0(X, nL) \text{ for all } n > 0\}.$$

- If $s \in H^0(X,L)$ then $Z(s) \subset X$ denotes the zero scheme of the section s.
- If $\alpha \in \mathbf{R}$ then $\lfloor \alpha \rfloor$ denotes the round–down of α, that is the largest integer less than or equal to α. Similarly, $\lceil \alpha \rceil$ denotes the round–up of α, the smallest integer greater than or equal to α.

1 The Three–fold Case

Before proving Theorem 0.2 we review the strategy of [EKL]. Suppose then that X is a smooth projective variety of dimension d and A an ample line bundle on X. Furthermore, let η be a very general point of X. Ein, Küchle, and Lazarsfeld study the linear series

$$\left| kA \otimes m_\eta^{k\alpha} \right|$$

for various values of α.

Roughly speaking, the argument of [EKL] goes as follows. Suppose that $\epsilon(\eta, A) < \frac{1}{d}$ and let C_η be a curve with

$$\frac{A \cdot C}{\mathrm{mult}_\eta(C_\eta)} < \frac{1}{d}. \tag{1.1}$$

Moreover, we assume that C_η is chosen from a flat family $\mathcal{F} \subset X \times T$ defined over a smooth affine variety T of dimension d with a quasi–finite map $\phi : T \to X$ where

$$\frac{A \cdot C_t}{\mathrm{mult}_{\phi(t)}(C_t)} < \frac{1}{d}, \quad \forall t \in T.$$

Consider, then, for k sufficiently divisible, the linear series

$$\left| kA \otimes m_\eta^{k\alpha} \right|, \quad k\alpha \in \mathbf{Z}. \tag{1.2}$$

If $k\alpha > k\epsilon(\eta, A)$ then by 1.1 the curve C_η is in the base locus of this linear series. Using the fact that C_η moves in the family $\mathcal{F}$ in order to "differentiate in the parameter direction", Ein, Küchle, and Lazarsfeld then show that any divisor $D \in \left| kA \otimes m_\eta^{k\alpha} \right|$ vanishes along C_η to order at least $k\alpha - k\epsilon(\eta, A)$. In particular, taking $\alpha > 2\epsilon(\eta, A)$ in 1.2, we see that if $D \in \left| kA \otimes m_\eta^{k\alpha} \right|$ then D vanishes to order greater than $k\epsilon(\eta, A)$ along C_η and hence vanishes along all curves in $C_t \in \mathcal{F}$ with $\phi(t) \in C_\eta$. The next step in the argument is to show that a

subfamily of the curves $\{C_t\}_{\phi(t)\in C_\eta}$, defined over a constructible subset $W \subset \phi^{-1}(C_\eta)$, sweep out an irreducible surface $S_\eta \subset X$. The argument is then iterated and the base locus of

$$\left|kA \otimes m_\eta^{k\alpha}\right|, \quad \alpha > r\epsilon(\eta, A)$$

is shown to contain an irreducible subvariety of dimension r, swept out by an $r-1$ dimensional subfamily of $\mathcal{F}$. After iterating this argument d times, a contradiction is reached because the linear series

$$\left|kA \otimes m_\eta^{kd\epsilon(\eta,A)+1}\right|$$

is forced to be empty but by hypothesis $d\epsilon(\eta, A) < 1$ and a simple argument using the Riemann–Roch theorem yields the desired contradiction.

Fundamental to the argument of [EKL] is the following "differentiation" result:

Lemma 1.3 ([EKL] Proposition 2.3) *Suppose $\eta \in X$ is a very general point and let $W \subset X$ be an irreducible subvariety. Let $\pi : Y \to X$ be the blow–up of η with exceptional divisor E and let $\tilde{W}$ is the strict transform of W in Y. Write*

$$\alpha(W) = \inf_{\beta\in\mathbf{Q}}\{\tilde{W} \subset \mathrm{BS}(\pi^* A(-\beta E))\}.$$

Suppose $\gamma > \alpha(W)$ and $0 \neq s \in H^0\left(X, nA \otimes m_\eta^{n\gamma}\right)$. Then

$$\mathrm{ord}_W(s) \geq n\gamma - \lfloor \alpha(W)n + 1\rfloor.$$

We have stated Lemma 1.3 in the form in which it will be used. To obtain this version from [EKL] Proposition 2.3, let $\Gamma \subset X \times T$ be the graph of $\phi : T \to X$. Let $p_1 : X \times T \to X$ and $p_2 : X \times T \to T$ denote the projections to the first and second factors respectively. Consider

$$\mathrm{BS}\left(p_1^*(kA) \otimes \mathcal{I}_\Gamma^{\lfloor k\alpha(W)+1\rfloor}\right).$$

By hypothesis for all $k > 0$ these subschemes contain an irreducible component $Z_k \subset X\times T$ so that $Z_k \cap \pi_2^{-1}(t) \supset W$ for any t with $\phi(t) = \eta$. As $k \to \infty$ one obtains a fixed subscheme $Z \subset X \times T$ with $W \subset Z_t$ where $Z_t = Z \cap \pi_2^{-1}(t)$. According to [EKL] Proposition 2.3, any section $\sigma \in H^0\left(X, p_1^*(kA) \otimes \mathcal{I}_\Gamma^{k\gamma}\right)$ must vanish to order at least $k\gamma - \lfloor k\alpha(W)+1\rfloor$ along Z. Indeed, if not, after differentiating $k\gamma - \lfloor \alpha(W)+1\rfloor$ times we obtain $\sigma' \in H^0\left(X, p_1^*(kA) \otimes \mathcal{I}_\Gamma^{\lfloor k\alpha(W)+1\rfloor}\right)$ not vanishing along Z and this is a contradiction. Lemma 1.3 is the translation of this statement for the family Z to the fibres Z_t. Note that when applying Lemma 1.3 we often will assume, for simplicity, that

$$\mathrm{ord}_W(s) \geq n(\gamma - \alpha(W)) :$$

indeed, for the asymptotic estimate on jets, the round–down and 1 are irrelevant.

Proof of Theorem 0.2 Suppose that Theorem 0.2 were false and

$$\epsilon(\eta, A) < \frac{1}{2}.$$

Thus through a very general point $\eta \in X$ there is a curve C_η with $A \cdot C_\eta / \mathrm{mult}_\eta(C_\eta) < \frac{1}{2}$. Choosing a suitable family of such curves $\mathcal{F} \subset X \times T$ as above, we claim that there is an open set $U \subset X$ such that for each $x \in U$ there is an irreducible curve C_x satisfying $A \cdot C_x = p$ and $\mathrm{mult}_\eta(C_x) = q$ with $p/q < 1/2$. If $\epsilon(\eta, A) = p/q$ and there is a curve C_η through η with $\mathrm{mult}_\eta(C_\eta) = q$ and $A \cdot C_\eta = p$ then this is satisfied. If there were a Seshadri exceptional surface S at η, that is a surface with $\frac{\deg_A(S)}{\mathrm{mult}_\eta(S)} = \epsilon(\eta, A)$, then an immediate contradiction is obtained using Lemma 1.3: choose $2p/q < \gamma < 1$ and $0 \neq \sigma \in H^0\left(X, nA \otimes m_\eta^{n\gamma}/m_\eta^{n\gamma+1}\right)$. According to Lemma 1.3 $\mathrm{ord}_S(\sigma) \geq n\gamma - \lfloor pn/q + 1 \rfloor$. Since $\mathrm{mult}_\eta(S) \geq 2$ this is not possible for $n \gg 0$. Since there must either be a Seshadri exceptional curve or a Seshadri exceptional surface when $\epsilon(\eta, A) < 1$ we must have a Seshadri excptional curve C_η through η as desired.

The goal of the proof is to estimate

$$\lim_{n\to\infty} \frac{h^0\left(X, nA \otimes m_\eta^{\frac{3pn}{q}}\right)}{n^3}. \tag{1.4}$$

We will show that this limit is positive and then we have a contradiction from [EKL] which shows that the linear series $\left| nA \otimes m_\eta^{\frac{(3p+\alpha)n}{q}} \right|$ is empty for any $\alpha > 0$. Let $\pi : Y \to X$ be the blow–up of X at η with exceptional divisor E. Choose a rational number α and a large positive integer n with $n\alpha \in \mathbf{Z}$. Then we have

$$\begin{aligned} h^0(X, nA) \quad - \quad & h^0\left(X, nA \otimes m_\eta^{\alpha n}\right) \\ = \quad & \sum_{k=0}^{\alpha n - 1} \left(h^0\left(X, nA \otimes m_\eta^k\right) - h^0\left(X, nA \otimes m_\eta^{k+1}\right)\right) \\ = \quad & \sum_{k=0}^{\alpha n - 1} \left(h^0\left(Y, \pi^*(nA)(-kE)\right) - h^0\left(Y, \pi^*(nA)(-(k+1)E)\right)\right). \end{aligned} \tag{1.5}$$

We have $E \simeq \mathbf{P}^2$ and using 1.5 and the exact sequence

$$0 \to H^0\left(Y, \pi^*(nA)(-(k+1)E)\right) \to H^0\left(Y, \pi^*(nA)(-kE)\right) \to H^0\left(E, \pi^*(nA)(-kE)\right)$$

we find

$$h^0(X, nA) - h^0\left(X, nA \otimes m_\eta^{\alpha n}\right) = \sum_{k=0}^{\alpha n - 1} h_Y^0\left(\mathbf{P}^2, \mathcal{O}(k)\right) \tag{1.6}$$

where $h_Y^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)$ denotes the dimension of the subspace of $H^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)$ coming via restriction from $H^0\left(Y, \pi^*(nA)(-kE)\right)$. Our goal, then, is for each value of k, to bound $h_Y^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)$ from above.

We next define critical numbers where the base locus of $\left|kA \otimes m_\eta^{k\alpha}\right|$ is forced to jump for numerical reasons:

$$\alpha_1 = \frac{p}{q},$$
$$\alpha_3 = \frac{2p}{q}.$$

There is also a more subtle jumping value between α_1 and α_3, at least for q sufficiently large, for which we require an extra definition. Let $Z \subset \mathbf{P}(T_\eta(X)) = \mathbf{P}^2$ denote the zero–dimensional subscheme of degree q given by $T_\eta(C_\eta)$. Then one can define a Seshadri constant associated to Z as follows. Suppose $\psi : Y \to \mathbf{P}^2$ is a birational map with Y smooth and $\psi^{-1}(\mathcal{I}_Z) = \mathcal{O}_Y(-E)$. Then

$$\epsilon(Z, \mathcal{O}(1)) = \sup_r \{r \in \mathbf{Q}^+ : \psi^*(\mathcal{O}(1))(-rE) \text{ is nef}\}.$$

Then, as we will see below, the base locus of $\left|kA \otimes m_\eta^{k\alpha}\right|$ is forced to contain a surface as soon as $\alpha > \alpha_2$ where α_2 satisfies

$$\frac{\alpha_2 - p/q}{2\alpha_2} = \epsilon\left(T_\eta(C_\eta), \mathcal{O}_{\mathbf{P}(T_\eta(X))}(1)\right).$$

Note that a surface could enter the base locus of $\left|kA \otimes m_\eta^{k\alpha}\right|$ for $\alpha < \alpha_2$. The numbers we chose are the "worst case scenario," the case where the linear series $|kA|$ generates the *most possible* jets at η. If it generates fewer jets, the numbers in the argument only improve. We note here that the reader not interested in the counting details can skip the analysis involving α_2. Indeed, when we prove Theorem 0.2 below there is enough room in the estimates so that the key result is Lemma 1.12 which applies to the jet analysis once we have exceded α_3. We included a more complete analysis both in order to reveal the subtleties involved in counting and because in other cases the more detailed analysis may be required.

By Lemma 1.3 we know that any section of $\left|kA \otimes m_\eta^{k\beta}\right|$ for $\beta > \alpha_1$ must vanish along C_η to multiplicity at least $k(\beta - \alpha_1)$. Once $\beta > \alpha_2$ we claim that the base locus of $\left|kA \otimes m_\eta^{k\beta}\right|$ must contain a surface S which passes through η. Indeed, if not, then choose $s_1, s_2 \in \left|kA \otimes m_\eta^{k\beta}\right|$ so that $T_\eta(Z(s_1))$ and $T_\eta(Z(s_1))$ meet properly inside $T_\eta(X)$. By Lemma 1.3 we have $\operatorname{mult}_{C_\eta}(s_1) \geq k(\beta - \alpha_1)$ and $\operatorname{mult}_{C_\eta}(s_2) \geq k(\beta - \alpha_1)$. By Theorem A of [ELS], we have $f_1, f_2 \in \mathcal{I}_C^{\lfloor k(\beta-\alpha)/2\rfloor}$ where $\mathcal{I}_C$ is the ideal sheaf of C and f_1 and f_2 are local equations for s_1 and s_2. Let $\pi : Y \to X$ be the blow–up of X at η with exceptional divisor E. Let $D_1 = \pi^*(Z(s_1))(-k\beta E)|E$ and $D_2 = \pi^*(Z(s_2))(-k\beta E)|E$. We have $E \simeq \mathbf{P}^2$ and D_1, D_2 are curves of degree $k\beta$ meeting properly along $T_\eta(C_\eta)$, each with multiplicity at least $\lfloor k(\beta - \alpha_1)/2\rfloor$ along $T_\eta(C_\eta)$. Considering the pencil of divisors spanned by D_1 and D_2 shows that $\psi^*(\mathcal{O}(k\beta))(-\lfloor k(\beta - \alpha_1)/2\rfloor E)$ is nef where $\psi : Y \to \mathbf{P}^2$ is a resolution of $\mathcal{I}_Z$ as above. It follows that

$$\epsilon(Z, \mathcal{O}(1)) \geq \frac{\lfloor k(\beta - \alpha_1)/2\rfloor}{k\beta} > \frac{\alpha_2 - p/q}{2\alpha_2},$$

contradicting the definition of α_2. Using Lemma 1.3 again, we conclude that there is a surface $S \subset X$ such that for $\beta > \alpha_2$ any divisor $D \in \left| kA \otimes m_\eta^{k\beta} \right|$ must vanish along S to order at least $k(\beta - \alpha_2)$. Finally, let S_η be the surface swept out by $\{C_x\}_{x \in Z}$ with $Z \subset \phi^{-1}(C_\eta)$ the constructible subset considered above. By Lemma 1.3 any divisor $D \in \left| kA \otimes m_\eta^{k\beta} \right|$ must vanish along S_η to order at least $k(\beta - \frac{2p}{q})$.

We are now prepared to bound $h^0_Y(\mathbf{P}^2, \mathcal{O}(k))$ from above, using the information about the order of vanishing of each section of $H^0_Y(\mathbf{P}^2, \mathcal{O}(k))$ along $T_\eta(C_\eta)$, $T_\eta(S)$, and $T_\eta(S_\eta)$. We will divide the estimate up into four cases

$$\begin{aligned} 0 \le k &\le n\alpha_1, \\ n\alpha_1 < k &\le n\alpha_2, \\ n\alpha_2 < k &\le n\alpha_3, \\ n\alpha_3 < k &\le \frac{3np}{q}. \end{aligned}$$

We assume for simplicity that $\alpha_i \in \mathbf{Q}$ and $n\alpha_i \in \mathbf{Z}$. For those α_i which are irrational, it suffices in the argument below to replace $n\alpha_i$ by $\lfloor n\alpha \rfloor$. Note also that if $\alpha_2 > \alpha_3$, one simply eliminates the third interval, replacing α_2 by α_3 in the second interval.

For small values of k one expects $|nA|$ to generate all k–jets and the estimate is

$$h^0_Y\left(\mathbf{P}^2, \mathcal{O}(k)\right) \le \binom{k+2}{2}, \quad 0 \le k \le n\alpha_1. \tag{1.7}$$

Next, for $n\alpha_1 < k \le n\alpha_2$ any section $\sigma \in H^0_Y(\mathbf{P}^2, \mathcal{O}(k))$ vanishes to order at least $\lfloor (k - n\alpha_1)/2 \rfloor$ along $T_\eta(C_\eta) \subset \mathbf{P}(T_\eta(X))$ giving the estimate

$$h^0_Y\left(\mathbf{P}^2, \mathcal{O}(k)\right) \le \binom{k+2}{2} - q\binom{\lfloor (k-n\alpha_1)/2 \rfloor + 1}{2} + o(k^2), \quad n\alpha_1 < k \le n\alpha_2. \tag{1.8}$$

This is established in Lemma 1.13 below.

Next suppose $n\alpha_2 < k \le n\alpha_3$. Let $\sigma \in H^0\left(X, nA \otimes m_\eta^k\right)$. For n suitably divisible, write

$$Z(\sigma) = aS + S', \text{ with } \operatorname{mult}_\eta(S') = n\alpha_2 :$$

this is possible since σ must vanish to order at least $k - n\alpha_2$ along the surface S. Let

$$\rho : H^0\left(X, nA(-aS) \otimes m_\eta^{n\alpha_2}\right) \to H^0(\mathbf{P}^2, \mathcal{O}(n\alpha_2))$$

be the restriction homomorphism. Then by definition we have

$$h^0_Y\left(\mathbf{P}^2, \mathcal{O}(k)\right) = \dim(\operatorname{Image}(\rho)).$$

Using the construction in [EKL] 3.8 we see that there exists an irreducible subvariety $V \subset X \times T$ such that $S = V \cap (X \times t)$ for some t with $\phi(t) = \eta$. In particular, since η is a very

general point it follows that there is a surface S' algebraically equivalent to S, not containing η, namely $S' = V \cap (X \times \xi)$ for a general point $\xi \in T$. Choose r sufficiently large so that $rA + b(S - S')$ is very ample for all $b > 0$. Choose $D \in |rA + a(S - S')|$ so that D does not contain η and let $E = D + aS'$. Then tensoring by E gives an injection

$$\rho_E : H^0\left(X, nA(-aS) \otimes m_\eta^{n\alpha_2}\right) \to H^0\left(X, (n+r)A \otimes m_\eta^{n\alpha_2}\right)$$

which preserves multiplicity at η. We conclude that

$$h_Y^0\left(\mathbf{P}^2, \mathcal{O}(k)\right) \le h^0\left(X, (n+r)A \otimes m_\eta^{n\alpha_2}/m_\eta^{n\alpha_2+1}\right), \quad n\alpha_2 < k \le n\alpha_3. \tag{1.9}$$

Finally suppose $n\alpha_3 < k \le \frac{3pn}{q}$. Suppose that $\mathrm{mult}_\eta(\sigma) = k$, $\sigma \in H^0(X, nA)$. We know from Lemma 1.3 that $\mathrm{mult}_{S_\eta}(\sigma) \ge k - n\alpha_3$. Since, according to Lemma 1.12 below $\mathrm{mult}_{C_\eta}(S_\eta) \ge 3$, we can write

$$Z(\sigma) = aS_\eta + S'$$

with $\mathrm{mult}_\eta(S') = k - 3(k - n\alpha_3)$. Arguing as in the previous case then gives

$$h_Y^0\left(\mathbf{P}^2, \mathcal{O}(k)\right) \le h^0\left(X, (n+r)A \otimes m_\eta^{3n\alpha_3-2k}/m_\eta^{3n\alpha_3-2k+1}\right), \quad n\alpha_3 < k \le \frac{3pn}{q}. \tag{1.10}$$

We are now prepared to evaluate the limit 1.4 using 1.6. We assume to begin with that $q \ge 5$. In particular this means that $\epsilon\left(T_\eta(C_\eta), \mathcal{O}_{\mathbf{P}(T_\eta(X))}(1)\right) < 1/2$ and this guarantees that $\alpha_2 < \alpha_3$. We divide the sum into four ranges of k determined by our critical numbers $\alpha_1, \alpha_2, \alpha_3$. First, by 1.7

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n\alpha_1} h_Y^0(\mathbf{P}^2, \mathcal{O}(k))}{n^3} \le \frac{\alpha_1^3}{6}.$$

Next, using 1.8 we see that

$$\lim_{n\to\infty} \frac{\sum_{k=n\alpha_1+1}^{n\alpha_2} h_Y^0(\mathbf{P}^2, \mathcal{O}(k))}{n^3} \le \frac{\alpha_2^3 - \alpha_1^3}{6} - \frac{q(\alpha_2 - \alpha_1)^3}{24}.$$

Using 1.8 and 1.9 gives

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{k=n\alpha_2+1}^{n\alpha_3} h_Y^0(\mathbf{P}^2, \mathcal{O}(k))}{n^3} &\le \lim_{n\to\infty} \frac{\sum_{k=n\alpha_2+1}^{n\alpha_3} h^0\left(X, (n+r)A \otimes m_\eta^{(n+r)(\frac{n\alpha_2}{n+r})}/m_\eta^{(n+r)(\frac{n\alpha_2}{n+r})+1}\right)}{n^3} \\
&\le (\alpha_3 - \alpha_2)\left(\frac{\alpha_2^2}{2} - \frac{q(\alpha_2 - \alpha_1)^2}{8}\right).
\end{aligned}$$

Finally, for $\sum_{k=n\alpha_3+1}^{3np/q} h_Y^0(\mathbf{P}^2, \mathcal{O}(k))$, using 1.10 and arguing as in the above case we can remove the r which is fixed. But the sum

$$\sum_{k=n\alpha_3+1}^{3np/q} h^0\left(X, nA \otimes m_\eta^{3n\alpha_3-2k}/m_\eta^{3n\alpha_3-2k+1}\right)$$

is simply every other term of the sum $\sum_{k=0}^{n\alpha_3} h^0_Y(\mathbf{P}^2, \mathcal{O}(k))$ and since our upper bound for $h^0_Y(\mathbf{P}^2, \mathcal{O}(k))$ varies as a piecewise polynomial this gives

$$\lim_{n\to\infty} \frac{\sum_{k=n\alpha_3+1}^{3np/q} h^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)}{n^3} \leq \frac{1}{2}\left(\frac{\alpha_2^3}{6} - \frac{q(\alpha_2-\alpha_1)^3}{24} + (\alpha_3-\alpha_2)\left(\frac{\alpha_2^2}{2} - q\frac{(\alpha_2-\alpha_1)^2}{8}\right)\right).$$

Combining all of the above estimates gives

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{k=0}^{3np/q} h^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)}{n^3} &\leq \frac{3}{2}\left(\frac{\alpha_2^3}{6} - \frac{q(\alpha_2-\alpha_1)^3}{24} + (\alpha_3-\alpha_2)\left(\frac{\alpha_2^2}{2} - q\frac{(\alpha_2-\alpha_1)^2}{8}\right)\right) \\ &\leq \frac{3}{2}\left(\frac{\alpha_2^3}{6} + (\alpha_3-\alpha_2)\left(\frac{\alpha_2^2}{2}\right)\right). \qquad (1.11)\end{aligned}$$

Note that in the last inequality we have omitted two of the negative or defect terms which were obtained by the detailed analysis above. The reason for this is that for q sufficiently large, the estimate 1.11 turns out to be sufficient to establish Theorem 0.2 while small values of q can be dealt with by hand. We included all of the counting details nonetheless as this is the technical heart of this paper.

In order to compute the upper bound in 1.11, we need to know the value of α_2. The Seshadri constant $\epsilon\left(T_\eta(C_\eta), \mathcal{O}_{\mathbf{P}(T_\eta(X))}(1)\right)$ is, however, very difficult to compute and so we look at the worst case scenario. In particular, the bound in 1.8 increases until $x = \frac{np}{q-4} + O(1)$ and then decreases. The bound in 1.9 then repeats the last value for the bound in 1.8 and then when one reaches 1.10 the values start to decrease. The $O(1)$ term will have no effect on the asymptotic estimate and thus the worst case to cosider is $\alpha_2 = \frac{np}{q-4}$. With this value of α_2 we need to assume that $q \geq 9$ in order to guarantee that $\alpha_2 < \alpha_3$. We find then, using 1.11,

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{k=0}^{3np/q} h^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)}{n^3} &\leq \frac{3}{2}\left(\frac{p^3}{6(q-4)^3} + \frac{p(q-8)}{q(q-4)}\left(\frac{p^2}{2(q-4)^2}\right)\right) \\ &= \frac{1}{6}\left(\frac{3p^3}{2(q-4)^2} + \frac{9p^3(q-8)}{2q(q-4)^3}\right)\end{aligned}$$

One checks that when $q \geq 10$ and $p/q < 1/2$ then

$$\frac{1}{6}\left(\frac{3p^3}{2(q-4)^2} + \frac{9p^3(q-8)}{2q(q-4)^3}\right) < \frac{1}{6}.$$

It follows from 1.6 that when $q \geq 10$

$$\lim_{n\to\infty} \frac{h^0\left(X, nA \otimes m_\eta^{\overline{\frac{3pn}{q}}}\right)}{n^3} > 0$$

and this concludes the proof of Theorem 0.2 when $q \geq 10$.

If $q < 10$ then there are only four possibilities which are not eliminated by [EKL], namely $p/q = 2/5, p/q = 3/7, p/q = 3/8$, and $p/q = 4/9$. We outline here how to eliminate the cases $p/q = 3/7$ and $p/q = 4/9$ which are the most difficult of the four. The counting here goes as follows. For $0 \leq k \leq np/q$ we use 1.7. For $np/q < k \leq 2np/q$, we use the estimate in 2.8 below. Finally for $2np/q < k \leq 3np/q$, we use 1.10. This gives, in the case where $p/q = 3/7$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{3np/q} h^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)}{n^3} \leq \frac{1}{6}\left(\frac{567}{686}\right).$$

For $p/q = 4/9$ we find

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{3np/q} h^0\left(\mathbf{P}^2, \mathcal{O}(k)\right)}{n^3} \leq \frac{1}{6}\left(\frac{224}{243}\right).$$

Lemma 1.12 *Suppose C_η satisfies*

$$\frac{A \cdot C_\eta}{\mathrm{mult}_\eta(C_\eta)} = \frac{p}{q} < \frac{1}{2}.$$

Let S_η be the surface swept out by $\{C_x\}_{x\in Z\subset\phi^{-1}(C_\eta)}$. Then

$$\mathrm{mult}_{C_\eta}(S_\eta) \geq 3.$$

Proof of Lemma 1.12 To see why S_η must be singular along C_η note that for a general point $\xi \in \phi(Z) \subset C_\eta$ there is a curve $C_\xi \subset S_\eta$ such that

$$\frac{A \cdot C_\xi}{\mathrm{mult}_\eta(C_\xi)} = \frac{p}{q} < \frac{1}{2}.$$

If S_η were smooth at a general point $\xi \in C_\eta$ then it would follow that $\epsilon(\xi, A|S_\eta) < \frac{1}{2}$ and this is impossible since Ein and Lazarsfeld [EL] have established that on a smooth surface the set of points where the Seshadri constant can be less than one is at most countable. In order to refine this argument, let $\pi : X' \to X$ be an embedded resolution of S_η. For $\xi \in C_\eta$ general let $\tilde{C}_\xi$ be the strict transform of C_ξ in X' and write

$$\pi^{-1}(\xi) \cap \tilde{C}_\xi = \{x_1, \ldots, x_r\}.$$

Suppose moreover that $\psi : C \to \tilde{C}_\xi$ is a desingularization with $\psi^{-1}(\pi^{-1}(\xi)) = \{y_1, \ldots, y_s\}$. Choose a linear series $|D|$ on X with D sufficiently positive so that if $E \in |D|$ is a general

member through ξ then $i(x_j, \tilde{C}_\xi \cdot \tilde{E} : X') = \text{mult}_{x_j}(\tilde{C}_\xi)$ for $1 \leq j \leq r$: this is possible by [F] 12.4.5. Then by [F] 12.4.5 and 7.1.17 we have

$$q = i(\xi, C_\xi \cdot D : X) = \sum_{j=1}^{s} \text{ord}_{y_j}(\psi^*(\pi^*(D)) = \sum_{i=1}^{r} \text{mult}_{x_i} \tilde{C}_\xi.$$

Now we have $s = \text{mult}_{C_\eta}(S_\eta)$ and thus if $s = 2$ we find that for $i = 1$ or $i = 2$

$$\frac{\pi^*(A) \cdot \tilde{C}_\xi}{\text{mult}_{x_i}(\tilde{C}_\xi)} < 1.$$

Let $\tilde{S} \subset X'$ be the resolution of S_η. The curves $\tilde{C}_\xi$ move in a one parameter family along the surface $\tilde{S}$ and thus $\{x \in \tilde{S} : \epsilon(x, \pi^*(A)) < 1\}$ is not countable, violating the main result of [EL]. Note that in [EL] Ein and Lazarsfeld state the main result for ample line bundles but the proof holds unchanged for a big and nef line bundle. Indeed, the only point where [EL] uses ampleness is to show that curves with bounded degree relative to the appropriate ample bundle A move in finitely many families but this also holds more generally when A is big and nef.

Lemma 1.13 *Suppose* $\text{mult}_\eta(C_\eta) = q$ *and* $n\alpha_1 < k \leq n\alpha_2$. *Then*

$$h^0_Y\left(\mathbf{P}^2, \mathcal{O}(k)\right) \leq \binom{k+2}{2} - q \binom{\lfloor (k - n\alpha_1 + 1)/2 \rfloor}{2} + o(k^2), \;\; n\alpha_1 < k \leq n\alpha_2.$$

Proof of Lemma 1.13 Let $A = \mathcal{O}_{\mathbf{P}^2}(1)$ where $\mathbf{P}^2 = \mathbf{P}(T_\eta(X))$ and let $Z \subset \mathbf{P}^2$ be the projectivized tangent cone of C_η at η. By definition of $\epsilon(Z, A)$, given $\delta > 0$ so that $\epsilon(Z, A) - \delta \in \mathbf{Q}$ for all $n > 0$ sufficiently large and divisible the evaluation map

$$H^0(\mathbf{P}^2, \mathcal{O}(n)) \to H^0\left(\mathbf{P}^2, \mathcal{O}(n) \otimes \mathcal{O}_{\mathbf{P}^2}/\mathcal{I}_Z^{n(\epsilon(Z,A)-\delta)}\right)$$

is surjective. According to [F] Example 4.3.4

$$\ell(\mathcal{O}_X/\mathcal{I}_Z^r) = \frac{qr^2}{2} + O(r).$$

In particular, for $n\alpha_1 < k \leq n\alpha_2$ we see that that Lemma 1.13 holds since any section of $H^0_Y(\mathbf{P}^2, \mathcal{O}(k))$ vanishes to order at least $\lfloor (k - n\alpha_1)/2 \rfloor$ along Z.

2 Counting jets in the general case

In addition to the computational complexity involved in estimating $h^0\left(X, nA \otimes m_\eta^k\right)$ for different values of k, the central difficulty in the higher dimensional case is Lemma 1.12 which uses the fact that on a surface X the set $\{x \in X : \epsilon(x, A) < 1\}$ is countable for an ample line bundle A. In particular, it is critical for Lemma 1.12 that this set contains no divisor and this is not known in higher dimension. In order to prove Theorem 0.3, we begin by recalling a key definition from [N].

Definition 2.1 *For an ample* $\mathbf{Q}$*–divisor* A *on a smooth surface* X *we let*

$$m(A) = \sup_{D \equiv A} \{\mathrm{mult}_\eta(D)\} \mid D \in \mathrm{Div}(X) \otimes \mathbf{Q} \textit{ effective}\} :$$

here $\equiv$ *denotes numerical equivalence.*

The importance of Definition 2.1 lies in the following simple result:

Lemma 2.2 *Suppose* X *is a projective variety of dimension* d *and* A *an ample line bundle on* X*. If*

$$\epsilon(\eta, A) < \frac{m(A)}{d}$$

then given $\delta > 0$ *there exists an irreducible proper subvariety* $Y \subset X$*, of dimension at least one, such that*

$$\epsilon(\xi, A|Y) < \epsilon(\eta, A) + \delta :$$

here ξ *is a very general point of* Y *and* $A|Y$ *is the restriction of* A *to* Y*.*

Proof of Lemma 2.2 Suppose, to the contrary, that for some $\delta > 0$

$$\epsilon(\xi, A|Y) \geq \epsilon(\eta, A) + \delta$$

for every irreducible $Y \subset X$. As above, following [EKL] 3.4, given $\epsilon > 0$ there is a a family of curves $\mathcal{F} \subset X \times T$ with

$$\alpha = \frac{A \cdot C_t}{\mathrm{mult}_{\phi(t)}(C_t)} < \epsilon(\eta, A) + \epsilon, \;\; t \in T. \tag{2.3}$$

This gives a chain of subvarieties

$$V_1 \subset V_2 \subset \cdots \subset V_d \tag{2.4}$$

where $V_1 = C_\eta$ for a very general point η and $V_{i+1} = C(V_i)$ in the notation of [EKL] Lemma 3.5.1: in particular, V_{i+1} is obtained by adjoining curves in the family $\mathcal{F}$. By [EKL] Lemma 3.5.1 each V_i is irreducible. According to Lemma 1.3 any section

$$s \in H^0\left(X, nA \otimes m_\eta^{2n\alpha+2}\right)$$

vanishes along C_η to order at least $n\alpha+1$ and hence vanishes along V_2. Proceding inductively using Lemma 1.3 we find that if $s \in H^0\left(X, nA \otimes m_\eta^{dn\alpha+d}\right)$ then

$$s|V_d = 0. \tag{2.5}$$

By hypothesis, $m(A) > d\epsilon(\eta, A)$ and thus, shrinking ϵ in 2.3 if necessary, we can assume that s is not indentically zero in 2.5 and thus $\dim(V_d) \leq d-1$. It follows from 2.4 that for some $1 \leq r \leq d-1$ we must have

$$V_r = V_{r+1}.$$

In particular for a general, hence smooth, point $\xi \in V_r$, we find a curve $C_\xi \subset V_r$ with

$$\frac{\mathrm{mult}_\xi(C_\xi)}{A \cdot C_\xi} = \alpha.$$

Hence

$$\epsilon(\xi, A|V_r) \leq \alpha < \epsilon(\eta, A) + \delta.$$

Thus we can take $Y = V_r$ and this proves Lemma 2.2.

We will derive Theorem 0.3 from Lemma 2.2 and the following result.

Lemma 2.6 *Suppose X is a projective variety of dimension $d \geq 4$ and A an ample line bundle on X. Then either*

$$m(A) > 1 + \frac{1}{3d}$$

or

$$\epsilon(\eta, A) > \frac{1}{d} + \frac{1}{3d^2}.$$

Proof of Lemma 2.6 The proof of Lemma 2.6 follows closely the method of §1 though the counting is much simpler. Let $\pi : Y \longrightarrow X$ be the blow–up of X at η with exceptional divisor $E \simeq \mathbf{P}^{d-1}$. Choose a rational number α, and a large positive integer n so that $n\alpha \in \mathbf{Z}$. Then we have, with the same notation as above

$$h^0(X, nA) - h^0\left(X, nA \otimes m_\eta^{\alpha n}\right) = \sum_{k=0}^{\alpha n-1} h_Y^0\left(\mathbf{P}^{d-1}, \mathcal{O}(k)\right) \tag{2.7}$$

where, as above, $h_Y^0\left(\mathbf{P}^{d-1}, \mathcal{O}(k)\right)$ denotes the dimension of the subspace of $H^0\left(\mathbf{P}^{d-1}, \mathcal{O}(k)\right)$ coming via restriction from $H^0\left(Y, \pi^*(nA)(-kE)\right)$.

Suppose that $x \in \mathbf{P}^{d-1} = \mathbf{P}(T_\eta(X))$ is a tangent vector to C_η at η. Then for $k > \epsilon(\eta, A)n$, Lemma 1.3 implies

$$h_Y^0\left(\mathbf{P}^{d-1}, \mathcal{O}(k)\right) \leq h^0\left(\mathbf{P}^{d-1}, \mathcal{O}(k) \otimes m_x^{\lceil k-\epsilon(\eta,A)n-1\rceil}\right) : \tag{2.8}$$

Combining 2.7 and 2.8 and taking the limit as $n \to \infty$ we find

$$\begin{aligned} \lim_{n\to\infty} \frac{h^0(X,nA) - h^0(X,nA\otimes m_\eta^{\alpha n})}{n^d} &\leq \frac{1}{(d-1)!}\int_0^\alpha \left(x^{d-1} - \max\left\{0,(x-\epsilon(\eta,A))^{d-1}\right\}\right) \\ &= \frac{\alpha^d - (\alpha-\epsilon(\eta,A))^d}{d!}. \end{aligned} \tag{2.9}$$

According to 2.9, if

$$\alpha^d - (\alpha - \epsilon(\eta,A))^d < 1$$

then $\lim_{n\to\infty} \frac{h^0\left(X,nA\otimes m_\eta^{\alpha n}\right)}{n^d} > 0$ and hence $m(A) > \alpha$. Suppose then that $\epsilon(\eta,A) \leq \frac{1}{d} + \frac{1}{3d^2}$ and let $\alpha = 1 + \frac{1}{3d}$. Then we find

$$\lim_{d\to\infty}\left(\left(\frac{3d+1}{3d}\right)^d - \left(\frac{3d+1}{3d} - \frac{3d+1}{3d^2}\right)^d\right) \leq e^{1/3} - e^{-2/3} < 0.9.$$

Thus we see that for all d sufficiently large $m(A) > 1 + \frac{1}{3d}$ if $\epsilon(\eta,A) \leq \frac{1}{d} + \frac{1}{3d^2}$. Elementary calculus suffices to show that this also holds for all $d \geq 4$ and this establishes Lemma 2.6.

Proof of Theorem 0.3 Suppose to the contrary that

$$\epsilon(\eta,A) \leq \frac{1}{d} + \frac{1}{3d^2}. \tag{2.10}$$

By Lemma 2.6 $m(A) > 1 + 1/3d$ and thus

$$\epsilon(\eta,A) < \frac{m(A)}{d}.$$

By Lemma 2.4 given $\delta > 0$ there is a proper subvariety $Y \subset X$ such that for a very general point $\xi \in Y$

$$\epsilon(\xi, A|Y) \leq \frac{1}{d} + \frac{1}{3d^2} + \delta.$$

But by the main theorem of [EKL], we know that $\epsilon(\xi,A|Y) \geq \frac{1}{\dim(Y)} \geq \frac{1}{d-1}$ and this is a contradiction for δ sufficiently small.

Note that above in 2.8 we have not counted carefully: in particular, the curve C_η is singular at η and thus the tangent space to C_η will be more than a single point with multiplicity one. Thus the counting can be improved considerably here but we were unable to obtain a significant quantitative improvement in the final result by checking this counting more carefully.

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
Electronic mail: nakamaye@math.unm.edu

References

[EL] L. Ein, R. Lazarsfeld, *Seshadri constants on smooth surfaces* Asterisque **218**, 1993, pp. 177–186.

[ELS] L. Ein, R. Lazarsfeld, and K. Smith, *Uniform bounds and symboic powers on smooth varieties*, Inv. Math., **144**, 2001, pp. 241–252.

[EKL] L. Ein, O. Küchle, R. Lazarsfeld, *Local positivity of ample line bundles*, J. Diff. Geomertry, **42**, 1995, pp. 193–219.

[F] W. Fulton, *Intersection Theory*, Springer, 1984.

[FW] G. Faltings and G. Wüstholz, *Diophantine approximations on projective spaces* , Inv. math., **116**, 1994, pp. 109–138.

[N] M. Nakamaye, *Seshadri constants and the geometry of surfaces*, to appear in Crelle, 2003.

[S] Y.–T. Siu, *Effective very ampleness*, Invent. Math., **124**, 1996, pp. 563–571.